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1990 J. Phys.: Condens. Matter 2 5277

(http://iopscience.iop.org/0953-8984/2/23/020)

J. Phys.: Condens. Matter **2** (1990) 5277–5282. Printed in the UK

LETTER TO THE EDITOR

The renormalisation of probability distributions in the random field problem

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Received 30 March 1990

Abstract. We study the Ising ferromagnet in a random field following the renormalisation of the distribution of random fields. The disorder in the local field induces fluctuations in the exchange couplings which are taken into account. Our method allows a straightforward determination of the exponents associated with the zero-temperature fixed point which governs the critical behaviour along the critical line and in particular the critical slowing down. We find the correlation length exponent $\nu = 1.0$ which is in very good agreement with numerical calculations. We also obtain the crossover exponent for small random fields.

The study of disordered magnetic systems has experienced enormous progress in this decade. Among these problems, that of a ferromagnet in a random field has reached a stage where, after much controversy, much of its behaviour is now understood [1]. It remained essentially a purely theoretical problem, formulated originally by Imry and Ma [2], until the work of Fishman and Aharony [3] showed that a diluted antiferromagnet in a uniform field provides its physical realisation. In the course of the theoretical efforts to understand the random field problem and in particular the question of its lower critical dimension, d_c, the failure of the renormalisation group ε expansion to provide the correct answers [4] came as a surprise. On the other hand, real space renormalisation group techniques have been applied successfully to the study of ferromagnets in random fields revealing its complex behaviour especially at very low temperatures [5]. More elaborate statistical treatment by Mckay and Berker [6], within the Migdal–Kadanoff approach, have been able to correctly reproduce the experimental results concerning, for example, the critical dimension. We should recall that early renormalisation group studies of spin glasses, using the Migdal–Kadanoff method, have correctly predicted the lower critical dimension for this intricate system [7] and so it is not so astonishing to find that in the random field case, this treatment also proves to be successful. In fact the study by Mckay and Berker [6] provides a nice realisation of the phase diagram for the Ising ferromagnet in a random field, for $d > d_c$, which was anticipated by Bray and Moore [8]. Also it brought up the concept of hybrid-order transitions which seem promising in reconciling controversy about the order of the phase transition of ferromagnets in random fields. It would be very interesting to check if this approach, in the case of large dimensions, will eventually yield the phase diagram obtained in the mean field approximation where the existence of a tricritical point has been demonstrated for a bimodal distribution [9].

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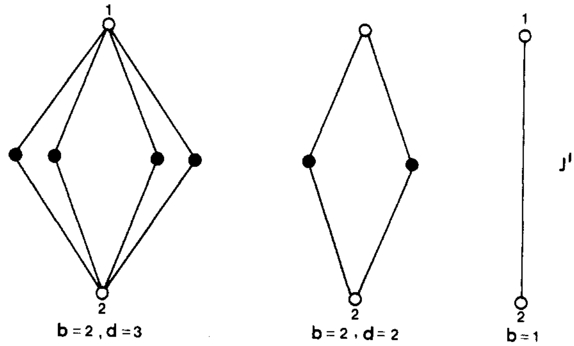

Figure 1. Hierarchical cells used in this work. 1 and 2 are terminal sites.

In this letter we study the Ising ferromagnet in a random field using the real space renormalisation group in hierarchical cells (see figure 1). We make use of the concept of transmissivity [10] to obtain recursion relations for the random field and the temperature or coupling constant. Our technique allows us to follow *separately* the evolution of the distribution of random fields and that of coupling constants along the renormalisation process. It differs consequently from that of [6] where the random fields at neighbouring sites and their connecting bonds are inextricably coupled together. Due to this separation, our approach provides the appropriate framework for a straightforward determination of the critical line and the critical exponents associated with the different fixed points, and the fixed-point form of the probability distributions. The notion of transmissivity [10] and its generalisation for the case where the system is under the influence of a magnetic field [11] has been extensively used to deal, successfully, with a variety of problems [5]. The basic idea is to introduce two auxiliary quantities, the transmissivities $t^+ = (Z^{++} - Z^{+-})/(Z^{++} + Z^{+-})$ and $t^- = (Z^{--} - Z^{-+})/(Z^{--} + Z^{-+})$ where $Z^{\alpha\beta}$ are the partition functions of the cells with the spins at the terminal sites (figure 1) kept fixed pointing up or down (+ or − respectively) in each case. We then use the property that the transmissivities remain invariant under a change in the length scale of the system to obtain recursion relations for the physical quantities of interest.

The Hamiltonian describing the system in each cell can be written in a convenient form as

$$H = \sum_{i,j} J_{ij}(1 - \delta_{\sigma_i\sigma_j}) + \sum_i (\boldsymbol{h}_i \boldsymbol{r}_i(1 - \delta_{\sigma_i r_i})) \tag{1}$$

where $\sigma_i = \pm 1$, J_{ij} are nearest neighbour ferromagnetic couplings, and $\boldsymbol{r}_i = \boldsymbol{h}_i/|\boldsymbol{h}_i|$. The $\boldsymbol{h}_i$ are random fields acting on the spins at sites i. Calculating the transmissivities t^+ and t^- for the cells with $b = 1$ and $b = 2$ shown in figure 1 and using their property of invariance under a change of length scale, we obtain recursion relations for the random fields and coupling constants of the form:

$$(\boldsymbol{h}_1/J)' = f[(\boldsymbol{h}_i/J_{ij}), (T/J_{ij})] \tag{2}$$

$$(T/J)' = g[(h_i/J_{ij}), (T/J_{ij})] \tag{3}$$

where the prime refers to the renormalised quantities in the smaller ($b = 1$) cell ($J_{ij} = J_{ji}$). We emphasise that a single renormalised field appears for the $b = 1$ cell since the other cancels out due to the definition of the transmissivities. The same occurs for the random fields acting on the spin in the equivalent terminal of the larger $b = 2$ cell. It is

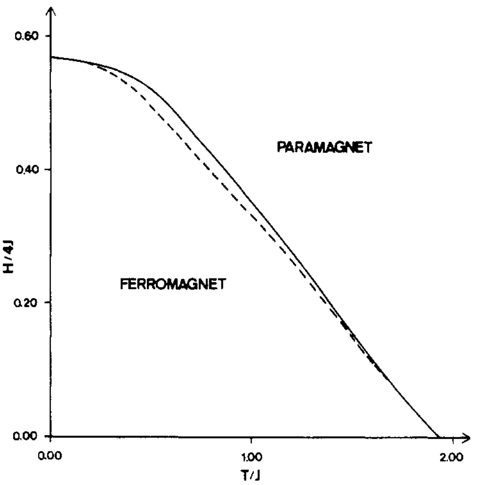

Figure 2. The phase diagram for $d = 3$ obtained for case 1 (---) and case 2 (——) (see text) with a Gaussian distribution. $H/4J$ represents $[\langle(h/4J)^2\rangle]^{1/2}$.

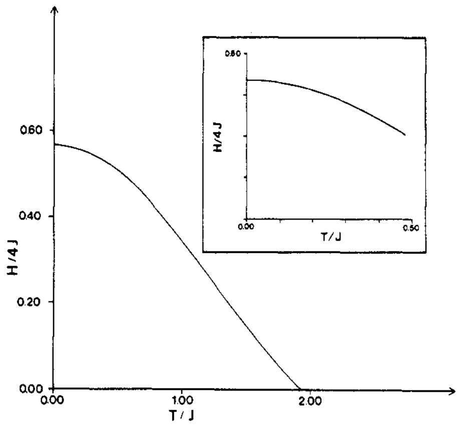

Figure 3. The phase diagram for the three-dimensional Ising model in a random field (case 2) with a Gaussian distribution. The inset shows the critical line at low temperatures described by the law $h_R^c(T) = h_R^c - a(T/J)^2$ with $h_R^c \approx 0.57$ and $a = 0.29$.

this cancellation which allows us to write the recursion relations in the simple form shown above. They express in a clear way the uncorrelated nature of the random fields acting in different sites. Because we have assigned two fields to each bond, the total field acting on a spin in a given site is a sum of z random fields where z is the number of bonds which converge to this site. This procedure, in the case of a uniform field, is equivalent to assigning a weight to the magnetic field acting on a spin equal to the coordination number of its site. It leads to the correct result that the uniform field scales like $\boldsymbol{h}_a' = b^d \boldsymbol{h}_a$ at the zero-temperature strong coupling attractor of the pure ferromagnet [5].

The renormalisation group equations shown above represent a system of coupled equations, and because the local fields are random the couplings also become random in the renormalisation process. We shall consider here the case where the couplings within a cell become random after the first iteration and also the simpler case of (2) and (3) where temperature fluctuations are taken into account for different cells of the ensemble of random field configurations but not within the cells. This amounts to defining all $J_{ij} = J$ in (2) and (3) above. For small random fields, and also at low temperatures, this represents a very good approximation since, starting with $J_{ij} = J$, the temperature or coupling constant distribution always remains very narrow compared to its average value in these ranges. The usefulness of this approximation can be appreciated in figure 2, where the phase diagram for $d = 3$ is calculated for constant J in the cell (case 1) and in the general case of random couplings within a cell (case 2). It can be seen that the phase diagrams are nearly the same except over a small temperature range.

The iteration of the recursion relations above do not generate negative temperatures or couplings so frustration is never introduced in the system. Also the distribution of coupling constants has a negligible weight at zero-J and consequently dilution does not play a role in this problem.

In order to iterate (2) and (3), we start with a given value of T/J and draw random fields $\boldsymbol{H} = \boldsymbol{h}/J$ from a distribution $P_h(H)$, with zero mean value and a given value for $\langle H^2 \rangle$. The random fields are fed into the right hand side of these equations to generate a new distribution $P'_h(H)$ of random fields and now also one of random couplings $P_J(T/J)$. This procedure is repeated a sufficient number of times and we follow what happens to these distributions during this process. The field distribution provides the relevant information to obtain the phase diagram. On the other hand the distribution of coupling constants yields the exponent, y, which enters in the modified hyperscaling relation as we discuss below.

In this letter we consider rectangular and Gaussian random field distributions which have a finite weight at the origin. Different distributions with a null probability of zero random field, like the bimodal distribution [9], may be in a different universality class and give rise to a tricritical point in the critical frontier. We observe two basic features in the course of the iteration of the probability distributions. Firstly there is an evolution in the shape of the original random field distributions. In the rectangular case, the distribution changes and approaches a Gaussian characterised by the ratio between the second and fourth moments ($\langle H^4 \rangle / \langle H^2 \rangle^2 = 3$ for a Gaussian) and visual observation of the computer screen. Also the width or second moment $\langle H^2 \rangle$ of this fixed point distribution, which is always centred in zero field ($\langle H \rangle = 0$), either increases or decreases due to the renormalisation. In the latter case the flow is towards the attractor of the ordered ferromagnetic phase characterised by $\langle T/J \rangle = \langle H^2 \rangle = 0$, where the averages $\langle \ldots \rangle$ are taken over the respective distributions. This corresponds to an ordered situation, that is to a random field distribution and temperature which are not sufficient to destroy the ferromagnetic phase. In the former case the flow is towards $\langle T/J \rangle = 0$ and $\langle H^2 \rangle = \infty$ which is clearly associated with the destruction of the ferromagnetic order. The line separating the basin of attraction of these two fixed points is the critical line that is shown in figure 3 for the three-dimensional case. This phase diagram was obtained by starting with a Gaussian distribution $P_h(H)$ to generate a new distribution with typically $N = 30\,000$ elements (so that $12 \times 30\,000$ draws were required, just for the random fields, at each step). To obtain the critical exponents, larger numbers of elements were considered ($N = 2 \times 10^5$). Along the critical line, the flow is towards the strong coupling attractor at $\langle T/J \rangle = 0$ and $\langle H^2 \rangle = \langle H^2 \rangle_c$ which is unstable at zero temperature. This instability is associated with an exponent ν which characterises the divergence of the correlation length ξ at zero temperature. That is, defining $h_R = (\langle H^2 \rangle)^{1/2}$ and $h_R^c = (\langle H^2 \rangle_c)^{1/2}$, we get $\xi \propto (h_R - h_R^c)^{-\nu}$ at $T = 0$ and $h_R \Rightarrow h_R^c$. We have found $\nu = 1.0$ which is in very good agreement with the numerical calculations of Ogielsky [12] which also yield $\nu = 1.0$. We point out that ours is not a strict $T = 0$ calculation, although we worked at very low temperatures ($T < T_c/40$) where ν has become 'saturated'. Also we considered the Gaussian as a fixed point distribution. Note that there is an error of at least $1/N^{1/2}$, where $N = 2 \times 10^5$, in our result.

The distribution of coupling constants, as mentioned before, always remains very narrow ($[\langle (T/J)^2 \rangle - \langle T/J \rangle^2]^{1/2} / \langle T/J \rangle \approx 10^{-3}$) for small random fields and low temperatures ($\approx 10^{-1}$). The sharpness of $P_J(T/J)$ can be appreciated from figure 2 where the critical line is shown for cases 1 and 2 (constant and random couplings in a cell respectively). These lines coincide, within numerical accuracy, except in the range $0.05 < \langle T/J \rangle < 0.6$, which is also the region where the scattering of the values of T/J around the mean value is larger. Note that the critical exponents calculated in both cases coincide.

We point out that for $d = 2$ we do not find a critical line and the flow is always towards the attractor of the disordered phase at $\langle T/J \rangle = 0$, $\langle H^2 \rangle = \infty$.

On the critical line, at very low temperatures, approaching the strong-coupling fixed point at $\langle H^2\rangle_c$, we find that temperature scales as $\langle T/J\rangle' = b^{-y}\langle T/J\rangle$ and consequently is an irrelevant 'field'. This irrelevance of temperature has been associated with the anomalous dynamics observed in random field systems [13]. In fact the exponent y controls the critical slowing down close to the transition line [13]. It is also responsible for the dimensional reduction which manifests itself for example in the modified hyperscaling relation [8] $\nu(d-y) = 2-\alpha$. Here d is the dimension of the system, ν and α are the correlation length exponent (defined before) and the specific heat exponent, respectively. Both are associated with the zero-temperature fixed point [8]. We can easily obtain this important exponent which we find to be $y = 1.48$. This value is consistent with an inequality predicted by Berker and McKay [14] for this problem. When we substitute the values for y and ν calculated above in the modified hyperscaling relation, we get $\alpha = 0.48$, which agrees with the existence of a second-order transition along the critical line.

We can also calculate how a uniform field $\boldsymbol{h}_a$ renormalises at the strong-coupling fixed point at $T=0$ and h_R^c essentially by considering a random field distribution with non-zero mean. We find $\boldsymbol{h}_a' = b^x \boldsymbol{h}_a$ with $x = 3.0$. The remarks concerning the evaluation of ν also hold here. Since the critical behaviour along the critical line is controlled by this zero-temperature fixed point, the fact that the exponent $x = 3$ (the dimension of the system) indicates that the magnetisation must be discontinuous when crossing the frontier in a finite random field [6]. This result together with the value of α obtained from the modified hyperscaling relation support the existence of an hybrid-order phase transition for the random field problem in three dimensions as suggested by Mckay and Berker [6].

Since we have found the Gaussian distribution of random fields to be a fixed point distribution, we can calculate the crossover exponent at small random fields and at T_c, the finite temperature transition of the pure ferromagnet. We find $\varphi = 2.0$, consistent with the linear increase of critical line at small random fields and the generalised scaling hypothesis [15] which implies $\psi = 2/\varphi$, where ψ is the shift exponent. Note that φ and ψ are defined here with respect to h_R such that the equation for the critical line is $T_c(h_R) = T_c - gh_R^{\psi}$ and the scaled field variable $h_R^2/|\varepsilon|^{\varphi}$. When comparing these results with experiment one should keep in mind that in real systems the observed crossover may be from random exchange to random field behaviour [1] and not that calculated above.

Finally we discuss the form of the critical line at very low temperatures. As shown in the inset of figure 3, this line is described by the expression $h_R^c(T) = h_R^c - aT^2$, so that the line is analytical close to $T=0$. As shown by Continentino and Oliveira [16], this is a consequence of the irrelevance of the temperature 'field'. In fact the scaling form of the correlation length [16] $\xi \propto (h_R - h_R^c)^{-\nu} f[T^2/(h_R - h_R^c), T^{1/\nu y}(h_R - h_R^c)]$. Although temperature may behave as a dangerously irrelevant variable in the random field problem, the shape of the critical line at low temperatures is dictated by the analytic temperature-dependent term of the scaling functions. We expect to find such analaytic temperature-dependent behaviour in other equilibrium properties of random field systems at very low temperatures.

To conclude, we have studied the ferromagnet in a random field in hierarchical lattices by accompanying the evolution of the probability distributions. Our results are in agreement with $d_c = 2$. We have obtained the exponents which control the critical behaviour in random fields and in particular the critical dynamics.

We would like to thank Drs S Mckay, S Galam, E Curado, P M Oliveira and Suzana M de Oliveira for useful discussions. We thank CNPq of Brasil for financial support.

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